Al'tshuler-Aronov correction to the conductivity of a large metallic square network

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We consider the correction $\Delta\sigma_{ee}$ due to electron-electron interaction to the conductivity of a weakly disordered metal (Al'tshuler-Aronov correction). The correction is related to the spectral determinant of the Laplace operator. The case of a large square metallic network is considered. The variation of $\Delta\sigma_{ee}(L_T)$ as a function of the thermal length L_T is found very similar to the variation of the weak localization $\Delta\sigma_{WL}(L_\varphi)$ as a function of the phase coherence length. Our result for $\Delta\sigma_{ee}$ interpolates between the known 1d and 2d results, but the interaction parameter entering the expression of $\Delta\sigma_{ee}$ keeps a 1d behaviour. Quite surprisingly, the result is very close to the 2d logarithmic behaviour already for $L_T \sim a/2$, where a is the lattice parameter.

I. INTRODUCTION

At low temperature, the classical (Drude) conductivity of a weakly disordered metal is affected by two kinds of quantum corrections : the first one is the *weak localization* (WL) correction, a phase coherent contribution that originates from quantum interferences between reversed electronic trajectories. This contribution to the averaged conductivity depends on the phase coherence length L_φ and the magnetic field : $\Delta\sigma_{WL}(\mathcal{B}, L_\varphi)$. The temperature manifests itself through L_φ, since phase breaking may depend on temperature, e.g. if it originates from electron-electron[1] or electron-phonon[2] interaction.

In a metal, an electron is not only elastically scattered on the disordered potential, but, due to the electron-electron interaction, is also scattered by the electrostatic potential created by the other electrons. At low temperatures, when the elastic scattering rate $(1/\tau_e)$ dominates the electron-electron scattering rate $(1/\tau_{ee}(T))$, the motion of the electron is diffusive between scattering events with other electrons. In this regime, electron-electron interaction is responsible for a small depletion of the density of states at Fermi energy (called the DoS anomaly or the Coulomb dip) and a correction to the averaged conductivity as well, the so-called *Al'tshuler-Aronov* (AA) correction[3,4,5,6,7,8,9] (see Refs.[10,11,12] for a recent discussion). AA and WL corrections are of the same order (but this latter vanishes in a magnetic field). However, contrary to the WL, the AA correction is not sensitive to phase coherence and involves another important length scale : the thermal length $L_T = \sqrt{D/T}$ ($\hbar = k_B = 1$). The AA correction, denoted below $\Delta\sigma_{ee}(L_T)$, has been measured in metallic wires in several experiments[14,15,16,17]. From the experimental point of view, AA correction allows to study interaction effects in weakly disordered metals, but also furnishes a local probe of temperature in order to control Joule heating effects[15,17], which is crucial in a phase coherent experiment.

All the works aforementioned refer to the quasi-one-dimensional (wire) or two-dimensional (plane) situations. Quantum transport has also been studied in more complex geometries like networks of quasi-1d wires. For example several studies of WL have been provided on large regular networks in honeycomb and square metallic networks[18,19], in square networks etched in a 2DEG[20], and in square and dice silver networks[21]. Theoretical studies of WL on networks have been initiated by the works of Douçot & Rammal[22,23] and improved by Pascaud & Montambaux[24] who introduced a powerful tool[25] : the spectral determinant of the Laplace operator, that will be used in the following (see also Ref.[26]).

The aim of this paper is to study how the AA correction can be computed in networks. In a first part we briefly recall how the spectral determinant can be used to compute the WL. Then in a second part we will consider the AA correction.

II. SPECTRAL DETERMINANT AND WEAK LOCALIZATION

Interferences of reversed electronic trajectories are encoded in the Cooperon, solution of a diffusion-like equation $(\partial_t - D[\nabla - 2ieA(x)]^2)\mathcal{P}_c(x, x'; t) = \delta(x - x')\delta(t)$, where $A(x)$ is the vector potential. On large regular networks, when it is justified to integrate uniformly the Cooperon over the network (see Ref.[27] for a discussion of this point) it is meaningful to introduce the space-averaged Cooperon $\mathcal{P}_c(t) = \int \frac{dx}{\text{Vol}} \mathcal{P}_c(x, x; t)$

then

$$\Delta\sigma_{\mathrm{WL}} = -\frac{2e^2 D}{\pi}\int_0^\infty \mathrm{d}t\, e^{-t/\tau_\varphi}\,\mathcal{P}_c(t) \quad (1)$$

$$= -\frac{2e^2}{\pi}\frac{1}{\mathrm{Vol}}\frac{\partial}{\partial\gamma}\ln S(\gamma) \quad (2)$$

where $\tau_\varphi = L_\varphi^2/D$ is the phase coherence time. The factor 2 stands for spin degeneracy. We have omitted in (1,2) a factor $1/s$ where s is the cross-section of the wires. The parameter γ is related to the phase coherence length $\gamma = 1/L_\varphi^2$ (note that description of the decoherence due to electron-electron interaction in networks requires a more refined discussion[28,29]). The spectral determinant of the Laplace operator is formally defined as $S(\gamma) = \det(\gamma - \Delta) = \prod_n(\gamma + E_n)$ where $\{E_n\}$ is the spectrum of $-\Delta$ [in the presence of a magnetic field, $\Delta \to (\nabla - 2ieA)^2$]. The interest in introducing $S(\gamma)$ is that it can be related to the determinant of a $V \times V$-matrix, where V is the number of vertices, that encodes all information on the network (topology, length of the wires, magnetic field, connection to reservoirs). We label vertices by greek letters. $l_{\alpha\beta}$ designates the length of the wire $(\alpha\beta)$ and $\theta_{\alpha\beta}$ the circulation of the vector potential along the wire. The topology is encoded in the adjacency matrix : $a_{\alpha\beta} = 1$ if α and β are linked by a wire, $a_{\alpha\beta} = 0$ otherwise. $\lambda_\alpha = \infty$ if α is connected to a reservoir and $\lambda_\alpha = 0$ if not. We introduce the matrix

$$\mathcal{M}_{\alpha\beta} = \delta_{\alpha\beta}\left(\lambda_\alpha + \sqrt{\gamma}\sum_\mu a_{\alpha\mu}\coth\sqrt{\gamma}l_{\alpha\mu}\right)$$
$$-a_{\alpha\beta}\sqrt{\gamma}\frac{e^{-i\theta_{\alpha\beta}}}{\sinh\sqrt{\gamma}l_{\alpha\beta}} \quad (3)$$

where the $a_{\alpha\mu}$ constraints the sum to run over neighbouring vertices. Then[24]

$$S(\gamma) = \prod_{(\alpha\beta)}\frac{\sinh\sqrt{\gamma}l_{\alpha\beta}}{\sqrt{\gamma}}\,\det\mathcal{M} \quad (4)$$

where the product runs over all wires. We now consider a large square network of size $N_x \times N_y$ made of wires of length $l_{\alpha\beta} = a\,\forall(\alpha\beta)$. For simplicity we impose periodic boundary conditions (topology of a torus), which is inessential as soon as the total size of the network remains small compared to L_φ. At zero magnetic field the spectrum of the adjacency matrix is $\epsilon_{n,m} = 2\cos(2n\pi/N_x) + 2\cos(2m\pi/N_y)$, with $n = 1,\cdots,N_x$ and $m = 1,\cdots,N_y$. Therefore

$$S(\gamma) = \left(2\frac{\sinh\sqrt{\gamma}a}{\sqrt{\gamma}}\right)^{N_x N_y} \quad (5)$$

$$\times \prod_{n=1}^{N_x}\prod_{m=1}^{N_y}\left(2\cosh\sqrt{\gamma}a - \cos\frac{2\pi n}{N_x} - \cos\frac{2\pi m}{N_y}\right).$$

The calculation of $\ln S(\gamma)$ involves a sum that can be replaced by an integral when N_x, $N_y \gg L_\varphi/a$. Using

$$\int_0^{2\pi}\frac{\mathrm{d}x\mathrm{d}y}{(2\pi)^2}\frac{1}{2A + \cos x + \cos y} = \frac{1}{\pi A}\,\mathrm{K}(1/A), \quad (6)$$

where $\mathrm{K}(x)$ is the complete elliptic integral of first kind[30], yields[20]

$$\frac{1}{\mathrm{Vol}}\frac{\partial}{\partial\gamma}\ln S(\gamma) = \frac{1}{4\sqrt{\gamma}}\left[\coth\sqrt{\gamma}a - \frac{1}{\sqrt{\gamma}a}\right.$$
$$\left.+\frac{2}{\pi}\tanh\sqrt{\gamma}a\,\mathrm{K}\left(\frac{1}{\cosh\sqrt{\gamma}a}\right)\right], (7)$$

where the volume of the network is $\mathrm{Vol} = 2N_x N_y a$. We recover the expression of the WL first derived by Douçot & Rammal[23]. Figure 1 displays the dependence of the WL correction as a function of the phase coherence length L_φ. We now discuss two limiting cases.
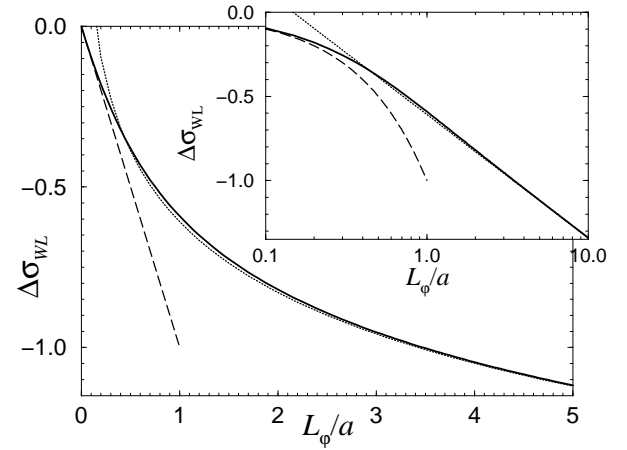

FIG. 1: $\Delta\sigma_{\mathrm{WL}}$ in unit of $2e^2/h$ as a function of L_φ/a (at zero magnetic field). The dashed line is the 1d result. The dotted line is the 2d limit eq. (10).

1d limit.– In the limit $L_\varphi \ll a$ (*i.e.* $\sqrt{\gamma}a \gg 1$) :

$$\Delta\sigma_{\mathrm{WL}} = -\frac{2e^2}{h}\left(L_\varphi - \frac{L_\varphi^2}{2a} + O\left(e^{-2a/L_\varphi}\right)\right) \quad (8)$$

We compare with the result for a wire of length a connected at its extremities : $\Delta\sigma_{\mathrm{WL}}^{\mathrm{wire}} \simeq -\frac{2e^2}{h}(L_\varphi - \frac{L_\varphi^2}{a})$. As we can see the dominant terms coincide. Deviations appear when L_φ/a increases since trajectories begin to feel the topology of the network. This is already visible by comparing the second terms of the expansions.

2d limit.– In the limit $L_\varphi \gg a$ (*i.e.* $\sqrt{\gamma}a \ll 1$), we

obtain

$$\frac{1}{\text{Vol}}\frac{\partial}{\partial\gamma}\ln S(\gamma) = \frac{a}{2\pi}\left[\ln(4L_\varphi/a) + \frac{\pi}{6}\right.$$
$$\left. +O\left(\frac{a^2}{L_\varphi^2}\ln\frac{L_\varphi}{a}\right)\right] \qquad (9)$$

The conductivity reads

$$\Delta\sigma_{\text{WL}} \simeq -\frac{2e^2}{h}a\left[\frac{1}{\pi}\ln(L_\varphi/a) + C_{\text{WL}}\right] \qquad (10)$$

with $C_{\text{WL}} = \frac{2\ln 2}{\pi} + \frac{1}{6} \simeq 0.608$. As noticed in the beginning of the section, eqs. (8,10) should be divided by the cross-section s of the wires. In the 2d limit, diffusive trajectories expand over distances larger than a and feel the two dimensional nature of the system, being the reason why (10) is reminiscent of the 2d result. It is interesting to point that the network provides a natural cutoff (the length of the wires, a) while the computation of the WL for a plane in the diffusion approximation requires to introduce a cutoff by hand for lower times in eq. (1), which is the elastic scattering time τ_e. In this latter case the constant added to the logarithmic behaviour is not well controlled since it depends on the cutoff procedure (the computation of the constant for a plane requires to go beyond the diffusion approximation and leads to[31] $\Delta\sigma_{\text{WL}}^{\text{plane}} = -\frac{e^2}{\pi h}\ln(2L_\varphi^2/\ell_e^2 + 1) \simeq -\frac{2e^2}{h}[\frac{1}{\pi}\ln(L_\varphi/\ell_e) + \frac{1}{2\pi}\ln 2]$ since $\ell_e \ll L_\varphi$).

III.   AL'TSHULER-ARONOV CORRECTION

At first order in the electron-electron interaction, the exchange term is the dominant contribution to the correction to the conductivity[8,10,11,32]

$$\Delta\sigma_{\text{ee}} = -\frac{2\sigma_0}{d\pi\text{Vol}}\int_{-\infty}^{+\infty}d\omega\frac{\partial}{\partial\omega}\left(\omega\coth\frac{\omega}{2T}\right)$$
$$\times\text{Im}\sum_{\vec{q}}D\vec{q}^2\frac{U(\vec{q},\omega)}{(-i\omega + D\vec{q}^2)^3} \qquad (11)$$

where $U(\vec{q},\omega)$ is the dynamically screened interaction. Within the RPA approximation and in the small $\vec{q}$ and ω limit, the interaction takes the form[33] $U(\vec{q},\omega) \simeq \frac{1}{2\rho_0}\frac{-i\omega + D\vec{q}^2}{D\vec{q}^2}$ where ρ_0 is the density of states per spin channel. Replacing the Drude conductivity by its expression $\sigma_0 = 2e^2\rho_0 D$ and performing an integration by parts, we get

$$\Delta\sigma_{\text{ee}} = -\frac{2e^2D}{\pi d\text{Vol}}\int d\omega\frac{\partial^2}{\partial\omega^2}\left(\omega\coth\frac{\omega}{2T}\right)$$
$$\times\text{Re}\sum_{\vec{q}}\frac{1}{-i\omega + D\vec{q}^2} \qquad (12)$$

After Fourier transform, the result can be cast in the form[11] :

$$\Delta\sigma_{\text{ee}} = -\lambda_\sigma\frac{e^2D}{\pi}\int_0^\infty dt\left(\frac{\pi Tt}{\sinh\pi Tt}\right)^2\mathcal{P}_d(t) , \qquad (13)$$

For the exchange term considered here, one finds $\lambda_\sigma = 4/d$. Further calculation yields[8] $\lambda_\sigma \simeq \frac{4}{d} - \frac{3}{2}F$, where F is the average of the interaction on the Fermi surface (see definition in Refs.[8,9]). This expression of λ_σ is valid in the perturbative regime, $F \ll 1$; nonperturbative expression is given in Refs.[6,7,8,9]. $\mathcal{P}_d(t)$ is the space integrated return probability $\mathcal{P}_d(t) = \int\frac{dx}{\text{Vol}}\mathcal{P}_d(x,x;t)$, where $\mathcal{P}_d(x,x';t)$ is solution of a classical diffusion equation similar to the equation for $\mathcal{P}_c(x,x';t)$, apart that it does not feel the magnetic field : $[\partial_t - D\Delta]\mathcal{P}_d(x,x';t) = \delta(x-x')\delta(t)$. Therefore the Laplace transform of $\mathcal{P}_d(t)$ is given by $\partial_\gamma\ln S(\gamma)$ with $\theta_{\alpha\beta} = 0$. It is interesting to point out that (13) has a similar structure to (1) with a different cutoff procedure for large time. It also involves a different scale : the temperature dependence of $\Delta\sigma_{\text{ee}}$ is driven by the length scale L_T instead of L_φ for the weak-localization correction $\Delta\sigma_{\text{WL}}$.

Up to eq. (13) the discussion is rather general and nothing has been specified on the system. We have seen in section II that the WL for the square network presents a dimensional crossover from 1d to 2d by tuning L_φ/a. A similar dimensional crossover occurs for the AA correction by tuning L_T/a as we will see. This remark raises the question of the dimension d in eq. (11). To answer this question we should return to the origin of the factor $1/d$: the current lines in the conductivity σ_{ij} produce a factor q_iq_j replaced by $\delta_{ij}\frac{1}{d}\vec{q}^2$ after angular integration. Since in a network the diffusion in the wires has a 1d structure (provided that $W \ll L_T \sim \sqrt{D/\omega}$, where W is the width of the wires), the dimension in λ_σ is $d = 1$. Therefore we have for the network $\lambda_\sigma^{\text{network}} \simeq 4 - \frac{3}{2}F$.

If one now expands the thermal function in (13) as :

$$\left(\frac{y}{\sinh y}\right)^2 = 4y^2\sum_{n=1}^\infty n\,e^{-2ny} , \qquad (14)$$

we can also relate $\Delta\sigma_{\text{ee}}$ to the spectral determinant. We obtain :

$$\Delta\sigma_{\text{ee}} = -\lambda_\sigma\frac{e^2}{\pi\text{Vol}}\sum_{n=1}^\infty\frac{1}{n}\left[\gamma^2\frac{\partial^3}{\partial\gamma^3}\ln S(\gamma)\right]_{\gamma=\frac{2n\pi}{L_T^2}} \qquad (15)$$

which is the central result of this paper. It is the starting point of the discussion below.

Application to the case of the square network.– We have to compute $\gamma^2\frac{\partial^3}{\partial\gamma^3}\ln S(\gamma)$. We start from (7) and compute its second derivative. We obtain after

some algebra :

$$\Delta\sigma_{\text{ee}} = -\lambda_\sigma \frac{e^2}{h} \frac{a}{8} \sum_{n=1}^{\infty} \frac{1}{n} \varphi\left(\sqrt{2n\pi}\frac{a}{L_T}\right) \quad (16)$$

where the function $\varphi(x)$ is given by :

$$\varphi(x) = -\frac{8}{x^2} + \frac{2x\cosh x}{\sinh^3 x} + \frac{3}{\sinh^2 x} + \frac{3\coth x}{x}$$
$$+\frac{2}{\pi}\left\{\left[\frac{3\tanh x}{x} - 3\right] \text{K}\left(\frac{1}{\cosh x}\right) + \left[3 - \frac{2x}{\sinh 2x}\right]\text{E}\left(\frac{1}{\cosh x}\right)\right\} , \quad (17)$$

$\text{E}(x)$ being the complete elliptic integral of second kind[30]. The function $\varphi(x)$ is plotted in figure 2 and its limiting behaviours are easily obtained[30] :

$$\varphi(x) = \frac{4}{\pi} + O(x^2) \qquad \text{for } x \to 0 \quad (18)$$

$$= \frac{6}{x} - \frac{8}{x^2} + O(xe^{-2x}) \text{ for } x \to \infty \quad (19)$$

The L_T dependence of AA correction on a square network is displayed on figure 3, where we have plotted $\Delta\sigma_{\text{ee}}(L_T)$ given by eq. (16). The dimensional crossover now occurs by tuning the ratio L_T/a. We consider the two limits.

FIG. 3: *The continuous line is $\Delta\sigma_{\text{ee}}$ in unit of $\lambda_\sigma \frac{e^2}{h}$ as a function of L_T/a (the series (16) is computed numerically). The dashed line is the 1d limit, eq. (20), and the dotted curve is the 2d limit, eq. (21).*
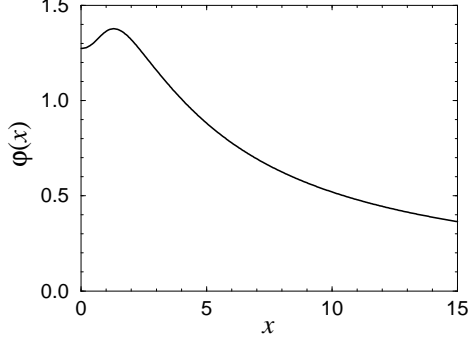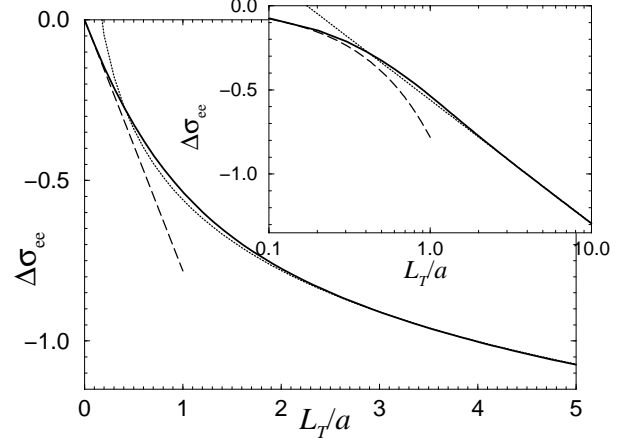

FIG. 2: *The function $\varphi(x)$ of eq. (17).*

1d limit.– For $L_T \ll a$ we can replace the expansion (19) in the series (16). Therefore

$$\Delta\sigma_{\text{ee}} \simeq -\lambda_\sigma \frac{e^2}{h}\left(\frac{3\zeta(3/2)}{4\sqrt{2\pi}} L_T - \frac{\pi}{12}\frac{L_T^2}{a}\right) \quad (20)$$

with $\frac{3\zeta(3/2)}{4\sqrt{2\pi}} \simeq 0.782$. The dominant term again coincides with the one for a connected wire[8,10,11] while the second differs by a factor 2, as for the WL [see discussion after eq. (8)].

2d limit.– In the limit $L_T \gg a$ we introduce $\mathcal{N} = (L_T/a)^2$ and cut the sum (16) in two pieces : $\sum_1^{\infty} = \sum_1^{\mathcal{N}} + \sum_{\mathcal{N}}^{\infty}$. It is clear from the limit behaviours of $\varphi(x)$ that the first sum diverges logarithmically with $\mathcal{N}$ while the second brings a negligible contribution of order $\mathcal{N}^0$. Therefore :

$$\Delta\sigma_{\text{ee}} \simeq -\lambda_\sigma \frac{e^2}{h} a \left[\frac{1}{\pi}\ln(L_T/a) + C_{\text{ee}}\right] \quad (21)$$

The constant is estimated numerically. We find $C_{\text{ee}} \simeq 0.56$.

The two eqs. (20,21) should be divided by the cross-section s of the wires.

The two functions $\Delta\sigma_{\text{WL}}(\mathcal{B} = 0, L_\varphi)$ (figure 1) and $\Delta\sigma_{\text{ee}}(L_T)$ (figure 3) are very similar. Apart from the prefactors $2e^2/h$ and $\lambda_\sigma e^2/h$ which account respectively for the spin degeneracy and the interaction strength, the linear behaviours at the origin have a different slope (1 and 0.782) and the logarithmic behaviours are slightly shifted : $C_{\text{WL}} \simeq 0.61$ and $C_{\text{ee}} \simeq 0.56$.

IV. COMPARISON WITH EXPERIMENTS

The AA correction has been recently measured by Mallet *et al*[34] in networks of silver wires with $3\,10^4$ and 10^5 cells, lattice spacing $a = 0.64\,\mu$m and diffusion constant $D \simeq 100\ \text{cm}^2/\text{s}$. The diffusion constant D has been measured separately (through measurement of the Drude conductance), therefore we can compare our result (16) with experiment using one fitting parameter only : the interaction parameter λ_σ. The 2d logarithmic behaviour (21) has been observed in the range $100\,\text{mK} < T < 1\,\text{K}$ from which the value $\lambda_\sigma^{\text{exp}} \simeq 3.1$ was extracted, in agreement with similar measurements performed on a long silver wire for which[34,35] $\lambda_\sigma^{\text{exp, wire}} \simeq 3.2$. We now compare with the theoretical value : for silver Fermi wavelength is $k_F^{-1} = 0.083$ nm and Thomas-Fermi screening length $\kappa^{-1} = 1/\sqrt{8\pi\rho_0 e^2} = 0.055$ nm. In the Thomas-Fermi approximation, the parameter F is given by[11] $F = (\frac{\kappa}{2k_F})^2 \ln[1 + (\frac{2k_F}{\kappa})^2]$, therefore $F \simeq 0.58$. Using the 1d nonperturbative expression[8] $\lambda_\sigma = 4 + \frac{48}{F}(\sqrt{1 + F/2} - 1 - F/4)$, we get $\lambda_\sigma^{\text{th}} \simeq 3.24$, close to the experimental value.

V. CONCLUSION

Equations (15,16) are our main results. The first one shows that AA and WL can be formally related :

$$\Delta\sigma_{\text{ee}}(L_T) = \frac{\lambda_\sigma}{2} \tag{22}$$
$$\times \sum_{n=1}^{\infty} \frac{1}{n} \left[\gamma^2 \frac{\partial^2}{\partial\gamma^2} \Delta\sigma_{\text{WL}}(L_\varphi) \right]_{\gamma \equiv \frac{1}{L_\varphi^2} = \frac{2n\pi}{L_T^2}}$$

The validity of this relation is the same as for eqs. (1,2) : the system should be such that it is meaningful to average uniformly the nonlocal conductivity $\sigma(r, r')$ to get the local conductivity $\sigma = \int \frac{\mathrm{d}r\mathrm{d}r'}{\text{Vol}} \sigma(r, r')$. A similar discussion has been proposed to relate WL and conductivity fluctuations (see appendix E of Ref.[29]).

Our starting point (11) is a formulation in the Fourier space, what implicitly assumes translation invariance. Whereas this assumption seems reasonable for a large regular network such as the square network studied in this article, its validity is not clear for networks of arbitrary topology, what would need further developments.

We have computed the AA correction in a large square network and shown that the result interpolates between the 1d, eq. (20), and a 2d result, eq. (21). Interestingly, the 2d limit in a network involves a 1d constant $\lambda_\sigma^{\text{network}} \simeq 4 - \frac{3}{2}F$, what is confirmed by experiments, as discussed in section IV.

The interest of the network compared to the plane is to control the constant C_{ee} of eq. (21) : for a plane, a cutoff must be introduced in eq. (13) at short time $t \sim \tau_e$ and the constant C_{ee} is replaced by a number that depends on the precise cutoff procedure. Experimentally, it would be interesting to observe the crossover from (20) to (21) by varying L_T/a. This was not possible in experiments of Mallet *et al*[34] described in section IV because measurements are complicated by the fact that electron-phonon interaction also brings a temperature-dependent contribution, $\Delta\sigma_{\text{e-ph}}$, at high temperature (above few Kelvins). The conductivity is given by $\sigma = \sigma_0 + \Delta\sigma_{\text{WL}} + \Delta\sigma_{\text{ee}} + \Delta\sigma_{\text{e-ph}}$. The WL can be suppressed by a magnetic field however the electron-phonon contribution is difficult to separate from $\Delta\sigma_{\text{ee}}$. Therefore the network should be patterned in a way such that the crossover 1d-2d remains below $T \sim 1$ K where $\Delta\sigma_{\text{e-ph}}$ is negligible. As an example we consider the silver networks studied in Ref.[21] for which $L_T = 0.27 \times T^{-1/2}$ (L_T in μm and T in K). In order to see clearly the 1d and the 2d regimes it would be convenient to study two networks with different lattice spacings. If temperature is constrained by $10\,\text{mK} < T < 1$ K, for $a = 0.5\,\mu$m we have $0.5 \lesssim L_T/a \lesssim 5$, which probes the 2d regime over one decade. A second lattice with $a \sim 5\,\mu$m would allow to probe the 1d regime since in this case $0.05 \lesssim L_T/a \lesssim 0.5$.

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